Nonlinear integral equations for the thermodynamics of the $sl(4)$-symmetric Uimin-Sutherland model

Jens Damerau and Andreas Klümpер

1 Abstract
We derive a finite set of nonlinear integral equations (NIE) for the thermodynamics of the $sl(4)$-symmetric Uimin-Sutherland (US) model. Our NIE can be numerically evaluated for arbitrary finite temperature and chemical potentials. In contrast to the NLE of type [2], which have already been generalised to $\mathbb{C}^2[\vartheta(t)]$, the evaluation of small temperatures poses no problem in our formulation. The known nonlinear integral equations for the sl(3) case [1] are recovered as a limiting case. We give numerical results for a spin-orbital model.

2 One-dimensional Uimin-Sutherland model
The Hamiltonian of the one-dimensional US model is given by
$$H = -\frac{\hbar^2}{2m}\sum_{i=1}^{N}\frac{\partial^2}{\partial x_i^2} + \sum_{i<j}^{N} \mu_i \delta(x_i - x_j),$$
where we have defined $\hbar^2/2m = \tilde{\hbar}$. The local interaction operator $\mu_i \delta(x_i - x_j)$ prevents neighbouring spins on the lattice with respect to their grading.

$$\sum_{j \neq i} (a_j(a_0 \ldots a_{i-1} a_{i+1} \ldots a_N) - (-1)^{N+i-j} a_0 \ldots a_{i-1} a_{i+1} \ldots a_N),$$
where periodic boundary conditions are imposed. We have added external field terms $\mu_i$, where $\mu_i$ counts the number of particles of type $\alpha$ sitting on site $i$, and $a_\alpha$ is some general chemical potential. The model is known to be exactly solvable on the basis of the Yang-Baxter algebra. The classical counterpart is the rational limit of the two-dimensional Perk-Schultz model with Boltzmann weights $e^{\beta(E_0 - E_1)}$, where $E_0$ and $E_1$ are the ground and first excited state eigenvalues. We introduce the quantum transfer matrix (QTM),
$$Z = \text{Tr} e^{\beta H} = \sum_{x \in \mathbb{Z}} \prod_{k = 1}^{n} a_k(x) a_k(x + 1),$$
where $Q(x)$ is the Trotter number, and $a_k(x)$ is the $k$th component of the QTM at site $x$.

3 Nonlinear integral equations for the sl(4) case
In general the US model has $sl(n)$ symmetry. However NLE of type [1] were previously known only for $n = \alpha \leq 3$. Here we treat the $sl(4)$-symmetric US model, for which we have $q = 4$ and all $e_i = 1$. The crucial point for the derivation of the NLE is the knowledge of suitable auxiliary functions. For convenience, we use an abbreviated notation using Young Tableaux,
$$\lambda_i \rightarrow \lambda_i x_i.$$

For the first fundamental representation we define four auxiliary functions:
$$b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2, \quad b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2, \quad b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2, \quad b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2.$$

Finally the four auxiliary functions for the third fundamental representation are:
$$b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2, \quad b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2, \quad b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2, \quad b_i(x) = \lambda_i x_i - \lambda_i x_i + 1/2.$$

4 Analytical investigation of the sl(3) limit
We want to show, how our formulation (13)-(24) reduces to the known NLE for the sl(3) symmetric case by freezing out one of the states. We choose the state $\alpha = 4$ and accordingly treat the limit $\alpha \rightarrow 0$. We observe, that only seven of the auxiliary functions survive. We can regard $b_i(x) = b_i(x) = b_i(x) = b_i(x) = b_i(x) = b_i(x) = b_i(x)$ as $\lambda_i x_i - \lambda_i x_i + 1/2$. Using this information the equation for $b_i(x)$ linearises and can be solved analytically. Substituting this into our NLE, we are again left with a NLE of type (13) but with only six auxiliary functions belonging to the two fundamental representations of sl(3). Here we get the kernel matrix
$$K_i(x) = -\frac{K_j(x)}{K_j(x)} + e^{-\beta K_j(x)},$$
the kernels are given in Fourier space as

$$K_j(x) = e^{\beta K_j(x)}, \quad K_j(x) = e^{\beta K_j(x)} + e^{\beta K_j(x)}.$$

The large eigenvalue is given by
$$\ln \Lambda_{\max}(0) = \beta \left( 1 - \frac{1}{e^{\beta K_j(x)}} \right).$$

As expected, this is exactly the known NLE for the sl(3) symmetric case [1].

5 Numerical results for a spin-orbital model
As an application, we consider the Hamiltonian of a $SU(2) \times SU(2)$ spin-orbital model at the supersymmetric point,
$$H = -\sum_{j=1}^{2} (\delta S_j^x S_j^x + \delta S_j^y S_j^y + \delta S_j^z S_j^z) + \sum_{j=1}^{N} (\delta S_j^x + \delta S_j^y + \delta S_j^z).$$

We have allowed for an external magnetic field $h$, which couples to the spins and orbital pseudo-spins with Landaf constants $\gamma_1$ and $\gamma_2$, respectively. Clearly the Hamiltonian is equivalent to the sl(4) US Hamiltonian (1), if we set
$$\mu_1 = (\gamma_1 + \gamma_2) h/2, \quad \mu_2 = (\gamma_1 - \gamma_2) h/2, \quad \mu_3 = (\gamma_1 - \gamma_2) h/2.$$

In the four figures below, results are shown for the entropy $S$, specific heat $C$, magnetisation $M$ and magnetic susceptibility $\gamma$ in the case $\gamma_1 = 1, \gamma_2 = 0$ for various magnetic fields. There is a critical field, which is known to be $h_c = 2/2 \geq 1.99$. For $h > h_c$ all spins are fully polarised in the ground state.

References